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ABSTRACT

Foundations of the Rank-Dependent Probability Weighting Function

The psychological basis for rank-dependent probability weighting, and for an inverse-S probability weighting function (PWF) in particular, has often been questioned. I examine the existence and shape of the PWF in a model allowing for optimism/pessimism over probability distributions and for loss averse/gain loving stochastic reference dependence. I give commonly observed shapes of PWF a psychological interpretation. In particular, I establish a deep connection between two of the most established phenomena in decisionmaking: loss aversion and the inverse-S PWF: the former is a pre- condition for the latter.

JEL Classification: D81, D01

Keywords: probability weighting, rank dependent expected utility, loss aversion, reference dependence, optimism, pessimism

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1 Introduction

Rank-dependent expected utility (RDEU) is one of the most prominent alternatives to expected utility. Introduced by Quiggin (1982), RDEU preserves standard axioms of rationality, such as monotonicity, transitivity, stochastic dominance and the book-making principle of de Finetti (1937), yet is capable of accounting for most of the well-known violations of expected utility theory, including Allais' paradox (Quiggin, 1985; Segal, 1987), the common ratio effect, and the preference reversal effect (Karni and Safra, 1987). The psychological foundation of rank-dependent probability transformations has been questioned, however, with some suggesting that this concept is purely a technical tool with no intuitive or psychological content.[1] In this paper, therefore, I explore the psychological interpretation of RDEU.

The only difference between expected utility and RDEU is that the latter transforms the probability distribution according to a probability weighting function (PWF) before computing the expectation of utility. But why would a PWF exist, and what determines its shape? I explore two routes that address these questions. The first route draws on the idea that the PWF reflects a stable preference for optimism or pessimism over probability distributions in the same way that the utility function reflects stable preferences over monetary amounts (e.g., Quiggin, 1982). The second route is based on the idea that the PWF indirectly reflects the emotions associated with the anticipation of experiencing gains and losses (e.g., Brandstätter *et al.*, 2002).

For either such route to offer a satisfactory account of the shape of the PWF, it should account for two stylized facts that emerge from an abundant experimental literature. First, behavior in experimental contexts is frequently consistent with an inverse-S shape for the

[1] For mathematical axiomatizations of RDEU see Abdellaoui (2002) and the references therein. While important, such approaches do not constitute an intuitive psychological justification for rank-dependent probability weighting.

PWF (see, e.g., Table 1 in Booij *et al.* (2010) for a summary).[2] Second, there is significant heterogeneity across subjects, some of whose choices are incompatible with an inverse-S shaped PWF. Evidence consistent with a strictly convex PWF is relatively plentiful (Jullien and Salanié, 2000; Goeree *et al.*, 2002; Qiu and Steiger, 2011; Harrison *et al.*, 2010; van de Kuilen and Wakker, 2011; Krawczyk, 2015). In a typical experiment at least a minority of subjects will exhibit an S-shaped or strictly concave PWF (see, e.g., Hey and Orme, 1994; Birnbaum and Chavez, 1997; Humphrey and Verschoor, 2004; Blavatskyy, 2010).[3]

Meeting these two stylized facts transpires to defeat explanations stemming separately from the two routes discussed above. In particular, as I shall show in what follows, both approaches are compatible with the existence of a PWF, but cannot replicate an inverse-S shaped PWF.[4] Hence the lingering doubts over the meaningful psychological interpretation of RDEU. Accordingly, a key contribution of this study is to examine whether the *conjunction* of these routes can account for these phenomena. I consider a (composite) model of decisionmaking in which decisionmakers both have preferences for optimism/pessimism over probability distributions *and* experience gain-loss emotions arising from stochastic reference dependence à la Kőszegi and Rabin (2006, 2007).[5]

I characterize the underlying psychological traits of optimism/pessimism and loss aver-

[2]For complementary evidence from non-laboratory environments see, e.g., Polkovnichenko and Zhao (2013), Rieger *et al.* (2017), and the references therein.

[3]Given the heterogeneity in experimentally estimated PWFs across subjects I focus away from approaches that are compatible with only a single shape of the PWF. An example is Tversky and Kahneman's (1992) principle of *diminishing sensitivity*, according to which people become less sensitive to changes in probability as they move away from a reference point. In the probability domain, it is argued, the two endpoints, zero and one, serve as reference points, thereby generating an inverse-S shape for the PWF. Other axiomatic and psychophysical approaches that focus purely on an inverse-S shape for the PWF include Luce (2001) and Takahashi (2011). These studies yield not only an inverse-S shape for the PWF, but indeed a specific parameterization of this shape proposed in Prelec (1998).

[4]The disappointment aversion models of, e.g., Bell (1985) and Gul (1991) also admit an RDEU representation, but only for the special case of binary lotteries (see, e.g., Abdellaoui and Bleichrodt, 2007). Moreover, the resulting PWF cannot take an inverse-S shape.

[5]A related composite model is discussed in the context of empirical applications in Barseghyan *et al.* (2013, 2018), but its potential to offer psychological foundations for RDEU has yet to be explored carefully.

sion/gain lovingness that give rise to particular shapes of PWF. A key finding in this context is that the model relates the inverse-S PWF to a unique psychological profile: necessary conditions for the PWF to be inverse-S shaped are that the decisionmaker is (i) optimistic (i.e., displays optimism over probability distributions); and (ii) loss averse. In this way I relate two of the most central features of decisionmaking: loss aversion and the inverse-S PWF. When I represent preferences for optimism/pessimism with a simple power function, an inverse-S shape for the PWF emerges for low enough levels of optimism and high enough levels of loss aversion.

The plan of the paper is as follows. Section 2 develops the model, and Section 3 examines the properties of the composite PWF. In Section 4 I present a simulation of the model for a simple parameterization. Section 5 concludes, and Figure 1 appears at the very rear.

2 Model

To investigate the psychological foundations of RDEU I now construct a model composed of two readily interpretable psychological underpinnings: decisionmakers may be (i) optimistic or pessimistic in their preferences over probability distributions, and (ii) evaluate lotteries relative to a stochastic reference point. The model is constructed to admit specifically an RDEU representation in which both preferences over probability distributions and the effects of stochastic reference dependence are represented simultaneously by a composite PWF.

2.1 Preferences over Probabilities

Let X be a random variable taking values on a finite subset of the real numbers, $\{x_1, \ldots, x_n\}$, ordered such that $x_1 < x_2 < \ldots < x_n$. Each outcome x_i occurs with probability p_i. The decisionmaker's preferences over the cumulative distribution function of X are represented by an *optimism-pessimism function*, $\pi : [0, 1] \to [0, 1]$, a continuous and increasing function, satisfying $\pi(0) = 0$ and $\pi(1) = 1$. Following Quiggin (1982), to allow a role for π, I suppose that the decisionmaker weighs outcome i not by its objective probability, p_i, but rather by the decision weight of outcome i, as given by

$$\pi_i = \pi\left(\sum_{j=i}^{n} p_j\right) - \pi\left(\sum_{j=i+1}^{n} p_j\right) \qquad i \in \{1, \ldots, n\}. \tag{1}$$

The potential for π to represent *optimism/pessimism* over probability distributions was first suggested by Quiggin (1982). The idea is described in full in Yaari (1987) and Diecidue and Wakker (2001), among other studies. Optimism corresponds to the situation in which an improvement in the ranking position of outcome x_i (by lowering the probability $\sum_{j=i+1}^{n} p_j$ of receiving a better outcome) increases π_i. It is equivalent to requiring π to be concave. Similarly, pessimism is equivalent to requiring π to be convex. Accordingly, to give π its desired psychological interpretation as an optimism-pessimism function, I therefore assume that either (i) $\pi''(p) \leq 0$ for all $p \in [0, 1]$ ("pessimism"); or (ii) $\pi''(p) \geq 0$ for all $p \in [0, 1]$ ("optimism"). Note that these conditions rule out switches in the sign of π'' on the unit interval.

2.2 Stochastic Reference Dependence

Following Kőszegi and Rabin (2006) the utility of an outcome x is judged relative to a stochastic reference lottery R.[6] The reference lottery is a random variable taking values on a finite subset of the real numbers, $\{r_1, \ldots, r_m\}$, ordered such that $r_1 < r_2 < \ldots < r_m$. Each outcome r_i occurs with probability q_i. As I permit the decisionmaker to have preferences for optimism/pessimism over the distribution function of X, so I likewise allow preferences for optimism/pessimism over the distribution function of R. These preferences are represented by a function θ defined in an exactly analogous manner to π. The decision weight assigned

[6] For an excellent review of models of reference-dependent decisionmaking see O'Donoghue and Sprenger (2018).

to reference outcome r_i under θ is then

$$\theta_i = \theta \left(\sum_{j=i}^{m} p_j \right) - \theta \left(\sum_{j=i+1}^{m} p_j \right) \qquad i \in \{1, \ldots, m\}.$$

Various considerations will impel me to assume θ and π to be identical functions in much of the analysis, but permitting these functions to be distinct initially is instructive when seeking to disentangle features of the model.

To introduce reference-dependence into utility I write the utility of an outcome x as

$$u(x|R) = v(x) + \lambda(x|R), \tag{2}$$

where $v(x)$ is absolute utility and $\lambda(x|R)$ is "gain–loss utility." I assume $v : \mathbb{R} \to \mathbb{R}$ is a continuous and increasing function, unique up to a positive affine transformation. Gain-loss utility is specified to reflect the idea that an outcome x is compared to every outcome that might have occurred in the reference lottery R:

$$\lambda(x|R) = \sum_{i=1}^{m} \theta_i \mu(v(x) - v(r_i)),$$

where the function μ is Kőszegi-Rabin's "universal gain–loss function." As in many contexts, I adopt a piecewise-linear specification for μ:

$$\mu(z) = \begin{cases} z & \text{if } z \geq 0; \\ [1+\lambda] z & \text{if } z < 0; \end{cases} \qquad \lambda \in [-1, 1]. \tag{3}$$

When $\lambda > 0$ the specification of μ in (3) captures the psychological concept of loss aversion, according to which losses loom larger than equivalent gains (Kahneman and Tversky, 1979). When $\lambda \in (-1, 0]$ gain loving preferences are implied.

Let $\zeta(i) = \sum_{j=1}^{m} \mathbf{1}_{x_i \geq r_k}$ denote the number of distinct values of R that do not exceed x_i. The indicator function $\mathbf{1}_a$ is one when condition a is true, and zero otherwise. Then the utility

$u(x_i|R)$ in (2) writes in full as

$$u(x_i|R) = v(x_i) + \sum_{j=1}^{\zeta(i)} \theta_i [v(x_i) - v(r_j)] - [1+\lambda] \sum_{j=\zeta(i)}^{m} \theta_i [v(r_j) - v(x_i)]. \quad (4)$$

A key insight of Kőszegi and Rabin (2006) is to interpret the reference lottery R as being the decisionmaker's expectation over the lottery X. In their *choice-acclimating personal equilibrium* (Kőszegi and Rabin, 2007) they argue that if a decisionmaker commits to a lottery X well in advance of the resolution of uncertainty, then by the time the uncertainty is resolved the decisionmaker will have come to expect the lottery X, and thus it becomes the reference lottery around which gains and losses are defined. Hence R coincides with X. Under this interpretation of R, (4) becomes

$$u(x_i|X) = v(x_i) + \sum_{j=1}^{i} \theta_j [v(x_i) - v(x_j)] - [1+\lambda] \sum_{j=i}^{n} \theta_j [v(x_j) - v(x_i)]. \quad (5)$$

2.3 Lottery Evaluation

Putting together the two planks of the model embodied by π_i in (1) and $u(x_i|X)$ in (5), the ex ante evaluation of lottery X is therefore

$$U(X|X) = \sum_{i=1}^{n} \pi_i u(x_i|X).$$

$U(X|X)$ can be written more informatively as a weighted sum of the absolute utilities:

Lemma 1 *The ex ante evaluation of X is given by*

$$U(X|X) = \sum_{i=1}^{n} \left\{ \pi_i \left[2 + \lambda \sum_{j=i}^{n} \theta_j \right] - \theta_i \left[1 + \lambda \sum_{j=1}^{i} \pi_j \right] \right\} v(x_i).$$

Lemma 1 clarifies that the probability weight of an individual outcome utilizes the whole distribution of outcomes, and the rank of the outcome in the distribution, as proposed in

Quiggin (1982). In the next section I show there exists an exact RDEU representation for $U(X|X)$.

2.4 Rank-dependent Representation

To admit a RDEU representation, it must be possible to write $U(X|X)$ in the form

$$U(X|X) = \sum_{i=1}^{n} w_i v(x_i),$$

where

$$w_i = w\left(\sum_{j=i}^{n} p_j\right) - w\left(\sum_{j=i+1}^{n} p_j\right),$$

and w – a function I term the composite PWF – satisfies $w(0) = 0$ and $w(1) = 1$.

Proposition 1 $U(X|X)$ *admits an RDEU representation with a composite weighting function* w *of the form*

$$w(p) = [2\pi(p) - \theta(p)] - \lambda\theta(p)[1 - \pi(p)].$$

Proposition 1 generalizes results found in Delquié and Cillo (2006), who implicitly assume $\theta(p) = p$, and Masatlioglu and Raymond (2016) and Barseghyan *et al.* (2018), who implicitly assume $\theta(p) = \pi(p)$. To understand the separate roles of π and θ, I differentiate pointwise in $w(p)$ to obtain

$$\frac{\partial w}{\partial \pi} \geq 0; \qquad \frac{\partial w}{\partial \theta} \leq 0; \qquad \frac{\partial^2 w}{\partial \pi \partial \theta} = \lambda. \tag{6}$$

From (6) it is seen that π and θ enter the composite PWF with opposing signs. In particular, as θ enters w negatively, *optimism* in θ translates into *pessimism* in w. The cross partial derivative confirms that if gains and losses are weighted equally ($\lambda = 0$) then there is no interaction in the composite PWF between probability weighting in the choice and reference lotteries. In the presence of loss aversion ($\lambda > 0$), however, these probability weightings interact positively in the composite PWF, and negatively under gain loving preferences $(\lambda < 0)$.

Having clarified the separate roles of π and θ, it seems desirable to set these functions identical, in the absence of a compelling psychological reason for assuming some probability distributions would be weighted differently from others by the same decisionmaker. An alternative motivation for this step arises when, as is conventional, I require the normative property of stochastic dominance to hold:

Proposition 2 *$U(X|X)$ satisfies stochastic dominance for all $\{\lambda, p\}$ if and only if $\theta = \pi$. In this case $w(p)$ becomes*

$$w(p) = \pi(p) - \lambda\pi(p)[1 - \pi(p)]. \tag{7}$$

Using Proposition 2, it is now possible to show formally that, on their own, the two planks of our model cannot predict an inverse-S shaped PWF. In the absence of stochastic reference dependence $(\lambda = 0)$ I obtain $w(p) = \pi(p)$. As, however, $\pi(p)$ reflects a stable trait for optimism or pessimism, it cannot take an inverse-S shape.[7] In the absence of optimism/pessimism $(\pi(p) = p)$ the composite PWF reduces to the linear-quadratic function $w(p) = p - \lambda p[1 - p]$. In this case, the sign of $w''(p)$ is the sign of λ, so again the composite weighting function cannot take an inverse-S shape. Accordingly, in the next section I investigate whether and how combining these two psychological features enables an explanation of the inverse-S PWF.

3 Understanding the Composite PWF

I now analyze the possible shapes of the composite PWF in (7). Differentiating (7) I have

[7] Only under what Neilson (2003: 181) describes as a "rather strange pattern of optimism and pessimism" in which a decisionmaker may be at once both optimistic *and* pessimistic with respect to different outcomes in the same gamble can the inverse-S PWF be explained in terms of optimism and pessimism alone. For evidence that preferences for optimism/pessimism are stable and heritable traits see, e.g., Bates (2015).

$$w'(p) = [1-\lambda+2\lambda\pi(p)]\,\pi'(p)\,; \quad (8)$$

$$w''(p) = [1-\lambda+2\lambda\pi(p)]\,\pi''(p)+2\lambda\,[\pi'(p)]^2\,. \quad (9)$$

The common forms of the PWF observed experimentally, i.e., inverse-S, S, concave, and convex, may all be distinguished with respect to (8) and (9), when evaluated at the endpoints $p \in \{0,1\}$ as in Table 1:

	$w'(0)$	$w'(1)$	$w''(0)$	$w''(1)$
Inverse-S	>1	>1	<0	>0
S	<1	<1	>0	<0
Concave	≥ 1	≤ 1	≤ 0	≤ 0
Convex	≤ 1	≥ 1	≥ 0	≥ 0

Table 1: Characterizing the shapes of the composite PWF

With respect to the inverse-S PWF we then have:

Proposition 3 *If the composite PWF is inverse-S the decisionmaker is optimistic and loss averse.*

Proposition 3 associates the inverse-S PWF to a unique underlying psychological profile: to observe this shape of PWF loss aversion is a prerequisite feature of preferences. Alongside loss aversion, an inverse-S decisionmaker must also be optimistic. No other underlying psychology is consistent with an inverse-S PWF. Rabin (2000) has described loss aversion as "the most firmly established feature of risk preferences." Demonstrating a link between loss aversion and another much-replicated feature of decisionmaking, the inverse-S PWF, therefore unifies two central, and apparently disjoint, phenomena. Intuitively, to ensure concavity of $w(p)$ for p close to zero requires the concavity of π (optimism). To then also generate steepness of $w(p)$ for p close to one requires the additional property of loss aversion.

As an immediate corollary of Proposition 3, necessary conditions for an S-shaped PWF are simply the opposite characteristics, i.e., gain lovingness and pessimism.

4 Parameterization

To explore the insights of the model further, I explore the shape of the composite PWF in (7) for a simple exponential choice of optimism-pessimism function: $\pi(p) = p^{\gamma}$, $\gamma > 0$. Optimism in π corresponds to $\gamma < 1$ while pessimism corresponds to $\gamma > 1$. Figure 1 shows in $(\log \gamma, \lambda)$-space the parameter combinations that induce each shape of PWF in Table 1.[8] It is seen that an inverse-S PWF arises for loss aversion and optimism when the former effect is sufficiently strong and the latter effect is sufficiently weak. An S-shaped PWF is associated with sufficiently strong gain lovingness and sufficiently weak pessimism. Global convexity of the PWF, which is relatively frequently observed in experimental data, is associated with loss aversion and pessimism. Indeed, for this specification of $\pi(p)$ (though not in general) pessimism is a necessary condition for a convex PWF. Last, a concave PWF is associated with optimism and weak loss aversion (or outright gain lovingness). In this way, a meaningful psychological interpretation may be given to a wide range of commonly observed shapes of PWF observed in experimental data.

<Figure 1 here – see p. 17>

5 Conclusion

Although one of the most prominent alternatives to expected utility, rank-dependent expected utility has often been thought to lack psychological foundations. In this paper I

[8] The distribution of decisionmakers on this space may well not be uniform, however. In particular, experimental evidence would suggest that the distribution might be more dense in the region consistent with loss aversion (the upper half of Figure 1) than the region consistent with gain lovingness.

sought to provide an intuitive explanation for the existence and shape of the probability weighting function. My model combines optimistic/pessimistic preferences over probability distributions with loss averse/gain loving gain-loss preferences. My key finding is that the inverse-S probability weighting function occurs for a unique combination of these preferences, namely loss aversion and optimism. As such, I forge a link between the hitherto disjoint concepts of loss aversion and the inverse-S probability weighting function. Future research should seek to validate experimentally the model by correlating the features of preferences I identify to the shape of the observed PWF.

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Appendix

Proof of Lemma 1. Beginning from (5) I have

$$\begin{aligned}
u\left(X|X\right) &= \sum_{i=1}^{n}\pi_i\left\{v\left(x_i\right)+\sum_{j=1}^{i}\theta_j\left[v\left(x_i\right)-v\left(x_j\right)\right]-\left[1+\lambda\right]\sum_{j=i}^{n}\theta_j\left[v\left(x_j\right)-v\left(x_i\right)\right]\right\}\\
&= \sum_{i=1}^{n}\left\{\pi_i\left[1+\sum_{j=1}^{i-1}\theta_j+\left[1+\lambda\right]\sum_{j=i+1}^{n}\theta_j\right]-\theta_i\left[\sum_{j=i+1}^{n}\pi_j+\left[1+\lambda\right]\sum_{j=1}^{i-1}\pi_j\right]\right\}v\left(x_i\right)\\
&= \sum_{i=1}^{n}\left\{\pi_i\left[1+\sum_{j=1}^{i}\theta_j+\left[1+\lambda\right]\sum_{j=i}^{n}\theta_j\right]-\theta_i\left[\sum_{j=i}^{n}\pi_j+\left[1+\lambda\right]\sum_{j=1}^{i}\pi_j\right]\right\}v\left(x_i\right)\\
&= \sum_{i=1}^{n}\left\{\pi_i\left[2+\theta_i+\lambda\sum_{j=i}^{n}\theta_j\right]-\theta_i\left[1+\pi_i+\lambda\sum_{j=1}^{i}\pi_j\right]\right\}v\left(x_i\right)\\
&= \sum_{i=1}^{n}\left\{\pi_i\left[2+\lambda\sum_{j=i}^{n}\theta_j\right]-\theta_i\left[1+\lambda\sum_{j=1}^{i}\pi_j\right]\right\}v\left(x_i\right).
\end{aligned}$$

■

Proof of Proposition 1. Write $w(p)$ as $w(p)=w_1(p)+w_2(p)$, where $w_1(p)=2\pi(p)-\theta(p)$ and $w_2(p)=-\lambda\theta(p)\left[1-\pi(p)\right]$. So $w_i=w_i^1+w_i^2$. Then

$$w_i^1=w^1\left(\sum_{j=i}^{n}p_j\right)-w^1\left(\sum_{j=i+1}^{n}p_j\right)=2\pi_i-\theta_i \tag{A.1}$$

and

$$\begin{aligned}
w_i^2 &= w^2\left(\sum_{j=i}^{n}p_j\right)-w^2\left(\sum_{j=i+1}^{n}p_j\right)\\
&= -\lambda\left\{\left[1-\pi\left(\sum_{j=i}^{n}p_j\right)\right]\theta\left(\sum_{j=i}^{n}p_j\right)-\left[1-\pi\left(\sum_{j=i+1}^{n}p_j\right)\right]\theta\left(\sum_{j=i+1}^{n}p_j\right)\right\}.
\end{aligned}$$

Noting that

$$\pi\left(\sum_{j=i}^{n}p_j\right)=\sum_{j=i}^{n}\pi_j$$

I may rewrite w_i^2 as

$$
\begin{aligned}
w_i^2 &= -\lambda\left\{\left[1-\pi\left(\sum_{j=i}^{n} p_j\right)\right]\sum_{j=i}^{n}\theta_j - \left[1-\pi\left(\sum_{j=i+1}^{n} p_j\right)\right]\sum_{j=i+1}^{n}\theta_j\right\} \\
&= -\lambda\left\{\left[1-\pi\left(\sum_{j=i}^{n} p_j\right)\right]\sum_{j=i}^{n}\theta_j - \left[1-\pi\left(\sum_{j=i+1}^{n} p_j\right)\right]\left[-\theta_i+\sum_{j=i}^{n}\theta_j\right]\right\} \\
&= -\lambda\left[\theta_i - \pi_i\sum_{j=i}^{n}\theta_j - \theta_i\pi\left(\sum_{j=i+1}^{n} p_j\right)\right] \\
&= -\lambda\left[\theta_i - \pi_i\sum_{j=i}^{n}\theta_j - \theta_i\sum_{j=i+1}^{n}\pi_j\right] \\
&= -\lambda\left[\theta_i\sum_{j=1}^{i}\pi_j - \pi_i\sum_{j=i}^{n}\theta_j\right]. \qquad (A.2)
\end{aligned}
$$

So, combining (A.1) and (A.2),

$$
\begin{aligned}
w_i &= 2\pi_i - \theta_i - \lambda\left[\theta_i\sum_{j=1}^{i}\pi_j - \pi_i\sum_{j=i}^{n}\theta_j\right] \\
&= \pi_i\left[2+\lambda\sum_{j=i}^{n}\theta_j\right] - \theta_i\left[1+\lambda\sum_{j=1}^{i}\pi_j\right],
\end{aligned}
$$

which matches the expression in Lemma 1. ■

Proof of Proposition 2. Stochastic dominance is equivalent to $w'(p) > 0$. We first show that $\theta \neq \pi \Rightarrow w'(p) < 0$ for at least one p. Using (8) and (9) it is straightforward to obtain that

$$
\begin{aligned}
w'(0) &\gtrless 1 \Leftrightarrow \pi'(0) \gtrless \frac{1}{1-\lambda}; \qquad (A.3)\\
w'(1) &\gtrless 1 \Leftrightarrow \pi'(1) \gtrless \frac{1}{1+\lambda}; \qquad (A.4)
\end{aligned}
$$

and

$$
\begin{aligned}
w''(0) &\gtrless 0 \Leftrightarrow \frac{\pi''(0)}{\pi'(0)} \gtrless -\frac{2\lambda\pi'(0)}{1-\lambda}; \qquad (A.5)\\
w''(1) &\gtrless 0 \Leftrightarrow \frac{\pi''(1)}{\pi'(1)} \gtrless -\frac{2\lambda\pi'(1)}{1+\lambda}. \qquad (A.6)
\end{aligned}
$$

Suppose $\lambda = -1$ then, at $p = 1$ $w'(1) \geq 0$ requires $\pi'(1) \geq \theta'(1)$. As $\{\pi, \theta\}$ are smooth functions that can only be concave, convex, or both concave and convex, and which must respect $\pi(0) = \theta(0) = 0$ and $\pi(1) = \theta(1) = 1$, it holds that $\pi'(1) = \theta'(1)$ only if $\theta = \pi$.

Hence, for distinct $\{\pi, \theta\}$ $w'(1) \geq 0$ implies $\pi'(1) > \theta'(1)$. The aforementioned properties of $\{\pi, \theta\}$ also entail that the condition $\pi'(1) > \theta'(1)$ implies $\pi'(0) < \theta'(0)$. But at $p = 0$ and $\lambda = 1$ $w'(p) \geq 0$ requires $\pi'(0) > \theta'(0)$. Hence, any distinct $\{\pi, \theta\}$ that induces $w'(1) \geq 0$ for $\lambda = -1$ will induce $w'(0) < 0$ for $\lambda = 1$. Having shown $\theta \neq \pi \Rightarrow w'(p) < 0$ we complete the proof by showing the reverse implication $w'(p) < 0 \Rightarrow \theta \neq \pi$. Suppose $w'(p) < 0$ then $\pi'(p)/\theta'(p) < \frac{1+\lambda[1-\pi(p)]}{2+\lambda\theta(p)}$. But, if $\theta = \pi$ then $\pi'(p)/\theta'(p) = 1 \geq \frac{1+\lambda[1-\pi(p)]}{2+\lambda\pi(p)}$, which is a contradiction. ■

Proof of Proposition 3. The inverse-s is characterised by $w''(0) < 0$ and $w'(1) > 1$. From (A.5) I have that

$$w''(0) < 0 \Rightarrow \pi''(0) < 0 \Rightarrow \pi''(p) < 0,$$

where the last inference follows as $\pi''(p)$ cannot switch sign. Hence the decisionmaker is optimistic. Then, given that π is increasing and satisfies $\pi(0) = 0$ and $\pi(1) = 1$, it follows that $\pi''(p) < 0 \Rightarrow \pi'(1) < 1$. Also, from (A.4), $w'(1) > 1 \Leftrightarrow \pi'(1) > [1+\lambda]^{-1}$. Combining these two inequalities implies that $[1+\lambda]^{-1} < \pi'(1) < 1$. For this condition to hold a necessary condition is that $[1+\lambda]^{-1} < 1$, which holds if and only if $\lambda > 0$. Hence the decisionmaker is loss averse. ■

Figures

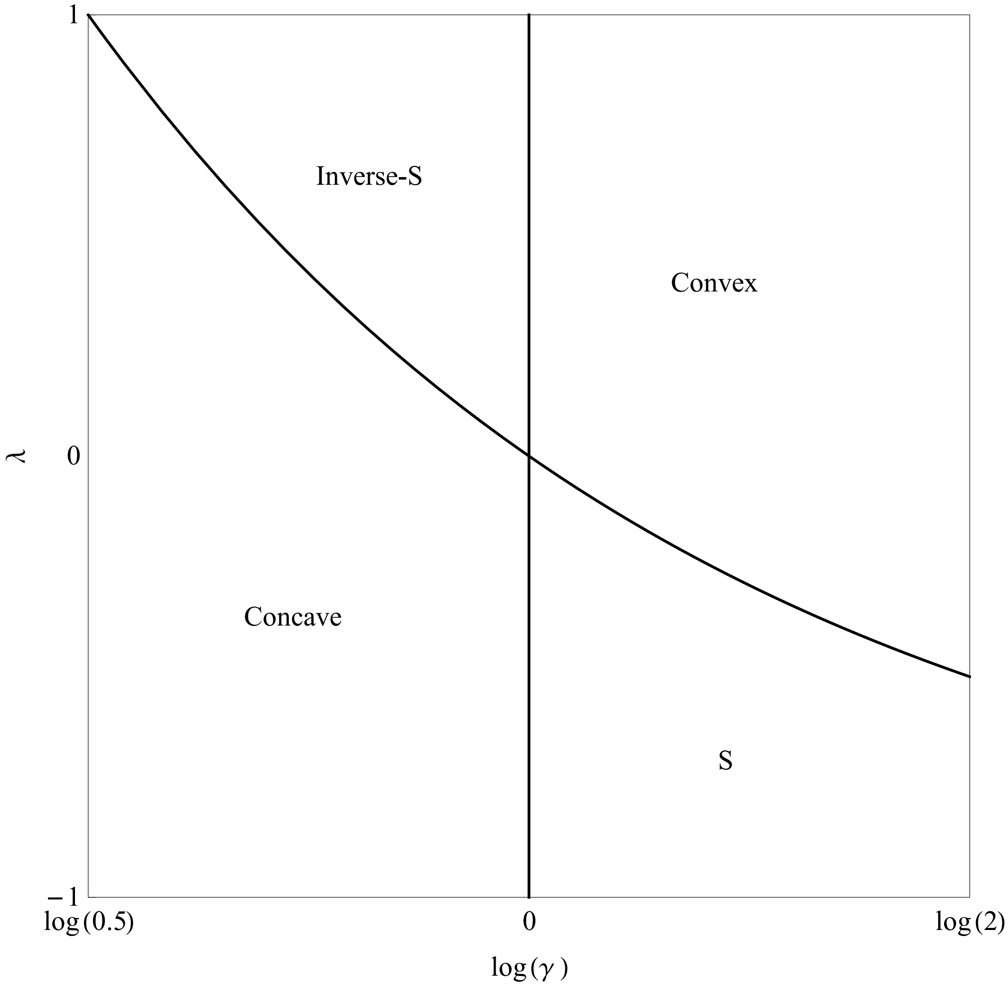

Figure 1: Shapes of the composite PWF in $(\log(\gamma), \lambda)$-space.