DISCUSSION PAPER SERIES

IZA DP No. 6589

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May 2012

Forschungsinstitut
zur Zukunft der Arbeit
Institute for the Study
of Labor

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Discussion Paper No. 6589
May 2012

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IZA Discussion Paper No. 6589
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ABSTRACT

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This paper is concerned with the problem of ranking and quantifying the extent of deprivation exhibited by multidimensional distributions, where the multiple attributes in which an individual can be deprived are represented by dichotomized variables. To this end we first aggregate deprivation for each individual into a "deprivation count", representing the number of dimensions for which the individual suffers from deprivation. Next, by drawing on the rank-dependent social evaluation framework that originates from Sen (1974) and Yaari (1988) the individual deprivation counts are aggregated into summary measures of deprivation, which prove to admit decomposition into the mean and the dispersion of the distribution of multiple deprivations. Moreover, second-degree upward and downward count distribution dominance are shown to be useful criteria for dividing the measures of deprivation into two separate subfamilies. To provide a normative justification of the dominance criteria we introduce alternative principles of association (correlation) rearrangements, where either the marginal deprivation distributions or the mean deprivation are assumed to be kept fixed.

JEL Classification: D31, D63, I32

Keywords: multidimensional deprivation, counting approach, partial orderings, rank-dependent measures of deprivation, principles of association rearrangements

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* We would like to thank Halvard Mehlum and Magne Mogstad for useful comments.

1. Introduction

Since the seminal papers of Sen (1976) and Foster-Greer-Thorbecke (1984), a flourishing literature has extended the normative approach of poverty measurement to the multidimensional case. In this paper we focus on multidimensional poverty measurement in situations where the multiple attributes in which an individual can be deprived are represented by dichotomized variables. This practice is conventionally adopted by statistical agencies, where information on whether people have income below a poverty threshold, suffer from poor health, lack social network, etc is collected (see e.g. Guio et al., 2009, or Alkire and Santos, 2010). The number of dimensions for which each individual suffers from deprivation may be summarised in a "deprivation count" (see Atkinson, 2003). Bossert et al. (2007) use the counting approach to analyse social exclusion in a dynamic context. Bossert et al. (2009), Lasso de La Vega and Urrutia (2011) and Alkire and Foster (2011) provide alternative axiomatic foundations of deprivation measures based on the counting approach.

Being deprived on a single dimension could result from the combination of a threshold and a continuous or discrete variable (e.g. income, or number of healthy days for year). In what follows it is supposed that available data only contain information on whether an individual is deprived or not on each dimension. This simplification allows us to delve into the question underlying the "identification" of the poor. Should we define poor only as those people suffering from deprivation on all dimensions or those that suffer from at least one dimension? These two opposite views correspond to the so-called "intersection" and "union" approaches in multidimensional poverty assessment. A related issue associated with multidimensional poverty analysis concerns the order in which the individual observations are aggregated (see Weymark 2006). Let us consider n individuals and r dimensions. Aggregating first individuals' deprivation on each dimension, the resulting indicators can be subsequently aggregated over the r dimensions generating an overall deprivation measure. The Human Poverty Index (HPI) is a prominent example of this approach.[1] By contrast, this paper relies on the opposite order of aggregation. First, by aggregating across the single dimensions for each individual a "deprivation count" is identified, representing the number of dimensions for which the individual suffers from deprivation. Second, an axiomatic approach is used to derive measures of deprivation that summarize the distribution of deprivation counts across individuals. As apposed to the HPI, this approach accounts for the association between deprivation indicators. Moreover, these measures of deprivation are shown to be decomposable with regard to the mean and the dispersion of deprivation counts, similarly as the mean-inequality trade-off for rank-dependent social welfare functions.

Atkinson (2003) investigated the relationship between expected utility type of summary measures of deprivation and the correlation between different attributes.[2] In the spirit of Bourguignon and Chakravarty (2003), Atkinson stressed the relevance of the sign of the cross derivatives of the individual "utility" function with respect to its arguments, and expressed doubts about the expected utility approach as the most attractive method for analysing counting data. By drawing on the rank-dependent theory of inequality measurement (Yaari 1987, Aaberge 2001) this paper introduces alternative ranking criteria for distributions of deprivation counts, where the conditions on the derivatives of the utility function arising from the expected utility model are shown to be replaced by simple conditions on a weight function used to distort probabilities in the rank-dependent framework. The shape of the weight function reveals whether the concern of the social planner is turned towards those

[1] See Anand and Sen (1997).
[2] See also Duclos et al. (2006).

people suffering from deprivation on all dimensions or those suffering from at least one dimension. This distinction is demonstrated also to be captured by two alternative partial orders; second-degree upward and downward count distribution dominance, which refines the trivial ranking imposed by Pareto dominance (or first-degree stochastic dominance) over the set of deprivation count distributions. We show that second-degree upward dominance generalizes the union approach whereas downward dominance generalizes the intersection approach. In order to provide a normative justification of the dominance criteria we introduce alternative principles of association (correlation) rearrangements, where either the marginal distributions or the mean deprivation are assumed to be kept fixed. The former case is analogue to the correlation-based rearrangement principles discussed in the literature (se e.g. Atkinson and Bourguignon, 1982, Bourguignon and Chakravarty, 2003 and Atkinson, 2003). However, as apposed to the previous literature we will make a distinction between whether an association rearrangement comes from a distribution characterized by positive or negative association between two or several deprivation indicators, in the spirit of the statistical literature on measurement of association in multidimensional contingency tables (formed by two or several dichotomous variables).The introduced association increasing/decreasing rearrangement principles will be proved to support second-degree downward/upward dominance under the condition of unchanged marginal distributions; i.e. the number of people suffering from each of the deprivation indicators are kept fixed. However, since real world interventions normally concern trade-offs that allow reduction in one deprivation indicator at the expense of a rise in another deprivation indicator, we find it attractive to introduce less restrictive association increasing/decreasing rearrangement principles that rely on the condition of fixed number of total deprivations rather than on the condition of keeping the number suffering from each of the indicators fixed.

The paper is organized as follows. Section 2 provides an axiomatic characterization of a family of deprivation measures. These deprivation measures generate linear orders on the set of deprivation count distributions and are shown to allow decomposition with regard to the extent and spread of deprivation counts. Second, to generalize the “union” and intersection” approaches, the criteria of second-degree upward and downward dominance are introduced. These criteria are shown to be equivalent to the intersection of linear orders on the set of deprivation count distributions generated by well-defined sets of deprivation measures. Section 3 discusses various association intervention principles linked to second-degree upward and downward dominance criteria and their relationship to two subfamilies of deprivation measures. Section 4 explains how the framework can be extended to account for different weights. A brief summary of the main results is given in Section 5. Proofs are gathered in the Appendix.

2. Ranking distributions of deprivation counts

We consider a situation where individuals might suffer from *r* different dimensions of deprivation. Let X_i be equal to 1 if an individual suffers from deprivation in the dimension *i* and 0 otherwise. Moreover, let $X = \sum_{i=1}^{r} X_i$ be a random variable with cumulative distribution function *F* and mean μ, and let F^{-1} denote the left inverse of *F*. Thus, $X = 1$ means that the individual suffers from one deprivation, $X = 2$ means that the individual suffers from two deprivations, etc. We call *X* the deprivation count. Furthermore, let $q_k = \Pr(X = k)$ which yields

(2.1) $$F(k)=\sum_{j=0}^{k} q_j,\ k=0,1,2...,r$$

and

(2.2) $$\mu=\sum_{k=1}^{r} kq_k\ .$$

Although F is a discrete distribution function we will for notational convenience occasionally use the integration symbol when we aggregate across count distributions.

Axiomatic justification of deprivation measures

Next, let $\boldsymbol{F}$ denotes the family of deprivation count distributions. A social planner's ranking over $\boldsymbol{F}$ can be represented by a preference relation $\succeq$, which will be assumed to satisfy the following axioms:

Axiom 1 (Order). $\succeq$ *Is transitive and complete over* $\boldsymbol{F}$.

Axiom 2 *(Continuity). For each* $F\in\boldsymbol{F}$ *the sets* $\left\{F^{*}\in\boldsymbol{F}:F\succeq F^{*}\right\}$ *and* $\left\{F^{*}\in\boldsymbol{F}:F^{*}\succeq F\right\}$ *are closed (w.r.t.* L_1*-norm).*

Axiom 3 *(First Stochastic Dominance FSD). Let* $F_1,F_2\in\boldsymbol{F}$ *. If* $F_1(k)\geq F_2(k)$ *for all* $k=0,1,2,...,r$ *then* $F_1\succeq F_2$.

Axiom 4 *(Dual Independence). Let* F_1*,* F_2 *and* F_3 *be members of* $\boldsymbol{F}$ *and let* $\alpha\in[0,1]$*. Then* $F_1\succeq F_2$ *implies* $\left(\alpha F_1^{-1}+(1-\alpha)F_3^{-1}\right)^{-1}\succeq\left(\alpha F_2^{-1}+(1-\alpha)F_3^{-1}\right)^{-1}$.

The first three axioms are quite conventional. Axiom 4 was introduced by Yaari (1987, 1988) as an alternative to the independence axiom of the expected utility theory. This axiom requires that the ordering of distributions is invariant with respect to certain changes in the distributions being compared. If F_1 is weakly preferred to F_2, then Axiom 4 states that any mixture on F_1^{-1} is weakly preferred to the corresponding mixture on F_2^{-1}. The intuition is that identical mixing interventions on the inverse distribution functions being compared do not affect the ranking of distributions.

To illustrate this averaging operation, let us consider the problem of evaluating the average deprivation within couples obtained by matching men and women with the same rank in the male and female deprivation count distributions (i.e. the most deprived man is matched with the most deprived woman, the second deprived man with the second deprived woman, and so on). Dual independence means that, given any initial distribution F_3 of deprivation over the female population, if within the male population, distribution F_1 is deemed to contain less deprivation than distribution F_2, this judgement is preserved after the matching with the women. Axiom 4 requires this property regardless of the initial patterns of deprivation and of the weights associated to male and female deprivation counts computing the average deprivation at the household level.

THEOREM 2.1. *A preference relation $\succeq$ on **F** satisfies Axioms 1-4 if and only if there exists a continuous and non-decreasing real function Γ defined on the unit interval, such that for all $F_1, F_2 \in$ **F***

$$F_1 \succeq F_2 \quad \Leftrightarrow \quad \sum_{k=0}^{r-1} \Gamma\Big(\sum_{j=0}^{k} q_{1j}\Big) \geq \sum_{k=0}^{r-1} \Gamma\Big(\sum_{j=0}^{k} q_{2j}\Big)$$

Where q_{ij}, with i=1,2 is the proportion of people with j deprivations in the two distributions, respectively. Moreover, Γ is unique up to a positive affine transformation.

For a proof of Theorem 2.1 we refer to Yaari (1987). Note, however, that Axiom 3 differs from the monotonicity axiom of Yaari (1987), which explains why Γ is non-decreasing.[3]

Summary measures of deprivation

Theorem 2.1 shows that a social planner who supports Axioms 1 – 4 will rank count distributions of deprivation according to the deprivation measure D_Γ defined by

(2.3) $$D_\Gamma(F) = r - \sum_{k=0}^{r-1} \Gamma\Big(\sum_{j=0}^{k} q_j\Big),$$

where Γ, with $\Gamma(0) = 0$ and $\Gamma(1) = 1$, is a non-decreasing function that represents the preferences of the social planner. Since *F* denotes the distribution of the deprivation count, $D_\Gamma(F)$ can be considered as a summary measure of deprivation exhibited by the distribution *F*. The social planner considers the distribution *F* that minimizes $D_\Gamma(F)$ to be the most favorable among those being compared.

Atkinson et al. (2002) and Atkinson (2003) call attention to the distinction between the union and intersection approaches for measuring deprivation. A social planner who supports the union approach is particularly concerned with the proportion of people who suffers from at least one dimension of deprivation $(1-q_0)$, whereas a social planner in favour of the intersection approach will focus attention on the proportion of people deprived on all dimensions (q_r). By choosing the following specification for Γ,

(2.4) $$\Gamma(t) = \begin{cases} 0 & if\ 0 \leq t < q_0 \\ q_0 & if\ t = q_0 \\ 1 & if\ q_0 < t \leq 1, \end{cases}$$

we get $D_\Gamma(F) = 1 - q_0$, which means that the union measure can be considered as a limiting case of the D_Γ-family of deprivation measures. The following alternative specification of the preference function,

(2.5) $$\Gamma(t) = \begin{cases} 0 & if\ 0 \leq t < 1-q_r \\ 1-q_r & if\ t = 1-q_r \\ 1 & if\ 1-q_r < t \leq 1, \end{cases}$$

[3] Since the ordering relation defined on the set of inverse distribution functions is equivalent to the ordering relation defined on ***F***, the proof of Theorem 2.1 might alternatively be derived from the proof of the expected utility theory for choice under uncertainty.

yields $D_\Gamma(F) = r - 1 + q_r$, which means that also the intersection measure represents a limiting case of the D_Γ-family of deprivation measures. Although the union and intersection measures do not belong to the D_Γ-family (which is generated by continuous Γ functions) these deprivation measures can be approximated within this class (see Le Breton and Peluso 2010 for general approximation results).

Partial orders
To deal with situations where deprivation count distributions intersect, weaker dominance criteria than first-degree dominance (Axiom 3) are called for. As will be demonstrated below it will be useful to make a distinction between aggregating across count distributions from below and from above. We first introduce the "second-degree downward dominance" criterion.[4]

DEFINITION 2.1A. *A deprivation count distribution F_1 is said to second-degree downward dominate a deprivation count distribution F_2 if*

$$\int_u^1 F_1^{-1}(t)dt \le \int_u^1 F_2^{-1}(t)dt \text{ for all } u \in [0,1]$$

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$.

A social planner who implements second-degree downward count distribution dominance is especially concerned about those people who suffer from deprivation over many dimensions. However, an alternative ranking criterion that focuses attention on those who suffer deprivation from few dimensions can be obtained by aggregating the deprivation count distribution from below.

DEFINITION 2.1B. *A deprivation count distribution F_1 is said to second-degree upward dominate a deprivation count distribution F_2 if*

$$\int_0^u F_1^{-1}(t)dt \le \int_0^u F_2^{-1}(t)dt \text{ for all } u \in [0,1],$$

and strict inequality holds strictly for some $u \in \langle 0,1 \rangle$.

Note that second-degree downward as well as upward count distribution dominance preserves first-degree dominance (Axiom 3) since first-degree dominance implies second-degree downward and upward dominance.

The following example illustrates the difference between the two principles: Consider two counting distributions F_1 and F_2. In distribution F_1 individual *i* suffers from *h* deprivations and individual *j* from *l* (*l*<*h*) deprivations. In distribution F_2 individual *i* suffers from *h+1* deprivations and individual *j* from *l-1* deprivations. The remaining individuals of the population have identical status in F_1 and F_2. A social planner who supports the condition of second-degree downward count distribution dominance will consider F_1 to be preferable to F_2. By contrast, a social planner who supports the condition of second-degree upward count distribution dominance will prefer F_2 to F_1. Accordingly, second-degree upward and

[4] Note that second-degree downward dominance is analogous to the notion of second-degree downward Lorenz dominance introduced by Aaberge (2009).

downward count distribution dominance might be considered as generalizations of the *union* and the *intersection* approach, respectively.

Let Ω_1 be a subset of the D_Γ-family, defined as follows

$$\Omega_1 = \{\Gamma : \Gamma'(t) > 0, \Gamma''(t) > 0 \text{ for all } t \in (0,1], \text{and } \Gamma'(0) = 0\}.$$

Note that $\Gamma'(0) = 0$ can be considered as a normalization condition. The following result provides a characterization of second-degree downward distribution dominance.

THEOREM 2.2A. *Let F_1 and F_2 be members of* $\mathbf{F}$. *Then the following statements are equivalent,*

(i) F_1 second-degree downward dominates F_2
(ii) $D_\Gamma(F_1) < D_\Gamma(F_2)$ *for all* $\Gamma \in \Omega_1$.

(Proof in Appendix).

To ensure equivalence between second-degree downward deprivation dominance and D_Γ-measures as ranking criteria, Theorem 2.2A shows that it is necessary to restrict the preference function Γ to be increasing and convex. If, by contrast, Γ is increasing and concave then Theorem 2.2B provides the analogy to Theorem 2.2A for upward dominance. Let Ω_2 be defined by

$$\Omega_2 = \{\Gamma : \Gamma'(t) > 0, \Gamma''(t) < 0 \text{ for } t \in \langle 0,1 \rangle, \text{and } \Gamma'(1) = 0\}.$$

THEOREM 2.2B. *Let F_1 and F_2 be members of* $\mathbf{F}$. *Then the following statements are equivalent,*

(i) F_1 second-degree upward dominates F_2
(ii) $D_\Gamma(F_1) < D_\Gamma(F_2)$ *for all* $\Gamma \in \Omega_2$.

(Proof in Appendix).

Note that even though members of Ω_1 and Ω_2 are strict convex and strict concave we will below denote them as convex and concave functions.

Decomposition of deprivation measures
As is well-known social welfare measures derived from the expected and rank-dependent utility theories allow multiplicative decompositions with regard to the mean and the inequality of income distributions (see Atkinson, 1970 and Yaari, 1988). An extension to the multidimensional case has been considered by Weymark (2006). In this section we show that the deprivation measures introduced above admit an analogous decomposition in terms of the mean and the dispersion of the deprivation count distributions. Moreover, it is demonstrated that the structure of this decomposition depends on whether the preferences of the social planner are more in line with the union or with the intersection approach.

The following example motivates the methods introduced in this section:

Example 1. Two alternative policies produce the following distributions of two-dimensional deprivation: F_1, where 50 per cent of the population suffers from one dimension and the remaining 50 per cent suffers from the other dimension; F_2 where 50 per cent of the population does not suffer from any deprivation and the remaining 50 per cent suffers from both dimensions. Thus, the mean number of deprivation is 1 for both distributions, but the intersection measure ranks F_1 to be preferable to F_2 whereas the union measure ranks F_2 to be preferable to F_1. An interesting question is which restrictions on Γ that guarantee that D_Γ ranks F_1 to be preferable to F_2 or vice versa.

As it will be demonstrated below, the ranking of F_1 and F_2 provided by D_Γ depends on whether Γ is convex or concave, which according to Theorems 2.2A and 2.2B depend on whether the social planner favors second-degree downward or upward count distribution dominance. This judgment can be equivalently expressed in terms of the mean and the dispersion of the deprivation count distributions. The intuition of this result is now presented through the two-dimensional case, then the general r-dimensional case follows.

Let $r = 2$, i.e. $X = X_1 + X_2$, and let

$p_{ij} = Pr\left(\left(X_1 = i\right) \cap \left(X_2 = j\right)\right)$, $p_{i+} = \Pr\left(X_1 = i\right)$, $p_{+j} = \Pr\left(X_2 = j\right)$.

Thus, $q_k = \Pr\left(X = k\right)$ can be expressed by $p_{ij}, i, j = 1,2$ in the following way:

$$q_0 = p_{00}$$
(2.6)
$$q_1 = p_{10} + p_{01}$$
$$q_2 = p_{11}.$$

The 2x2 case can be illustrated by the following table:

Table 2.1. The distribution of deprivation in two dimensions

		X_2		
		0	1	
X_1	0	p_{00}	p_{01}	p_{0+}
	1	p_{10}	p_{11}	p_{1+}
		p_{+0}	p_{+1}	1

The distribution *F* of *X* is given by

(2.7) $$F(k) = \Pr(X \le k) = \sum_{j=0}^{k} q_j, \; k = 0,1,2,$$

where $F(2) = 1$ and the mean $\mu = q_1 + 2q_2$.

In this case the class of deprivation measures $D_\Gamma(F)$ defined by (2.3) is given by

(2.8) $$D_\Gamma(F) = 2 - \Gamma(1 - q_2) - \Gamma(q_0).$$

Note that Γ can be interpreted as a preference function of a social planner that assigns lower weights for one than for two deprivation counts.

To supplement the information provided by $D_\Gamma(F)$ and μ, it will be useful to introduce the following measure of dispersion,

$$(2.9)\quad \Delta_\Gamma(F)=\begin{cases}\sum_{k=0}^{1}\left[\sum_{j=0}^{k}q_j-\Gamma(\sum_{j=0}^{k}q_j)\right]=q_0-\Gamma(q_0)+(1-q_2)-\Gamma(1-q_2) & \textit{when } \Gamma \textit{ is convex}\\ \sum_{k=0}^{1}\left[\sum_{j=0}^{k}\Gamma(\sum_{j=0}^{k}q_j)-\sum_{j=0}^{k}q_j\right]=\Gamma(q_0)-q_0+\Gamma(1-q_2)-(1-q_2) & \textit{when } \Gamma \textit{ is concave}\end{cases}$$

It can easily be observed from (2.9) that $\Delta_\Gamma(F)=0$ if and only if q_0, q_1 or q_2 is equal to 1, which means that every individual suffers from 0, 1 or 2 deprivations. Since $q_0+(1-q_2)=2-q_1-2q_2=2-\mu$, by inserting (2.9) in (2.8) it follows that the deprivation measure D_Γ admits the following decomposition

$$(2.10)\qquad D_\Gamma(F)=\begin{cases}\mu+\Delta_\Gamma(F) & \textit{when } \Gamma \textit{ is convex}\\ \mu-\Delta_\Gamma(F) & \textit{when } \Gamma \textit{ is concave.}\end{cases}$$

Thus, by using (2.10) we may identify the contribution to D_Γ from the average number of deprivations (μ) as well as from the dispersion of deprivations across the population. Moreover, expression (2.10) demonstrates that a social planner who is concerned about reducing the mean number of deprivations as well as the dispersion of deprivations across the population will use a measure D_Γ with a convex Γ whenever he/she pays particular attention to people who suffer from many deprivations. By contrast, when the social planner uses criterion D_Γ with a concave Γ, he/she is more concerned about the number of people who are deprived on at least one dimension (the union approach) as compared to individuals deprived on all dimensions (the intersection approach). In this case D_Γ can be expressed as the difference between the mean number of deprivations in the population and the dispersion of deprivations across the population. Thus, with Γ concave, D_Γ decreases when Δ_Γ increases.

By employing the criterion $D_\Gamma(F)$ defined by (2.10) to Example 1, it follows that F_1 is preferred if the social planner relies on a convex Γ. By contrast, F_2 is considered to be preferable if a concave Γ represent the preferences of the social planner.

By inserting for $\Gamma(t)=2t-t^2$ or $\Gamma(t)=t^2$ in (2.8) and (2.9) we get the following expressions for the Gini measure of deprivation and the Gini measure of dispersion (which corresponds to the Gini mean difference $\int F(x)(1-F(x))dx$)[5],

$$(2.11)\qquad D_\Gamma(F)=\begin{cases}\mu+q_1(1-q_1)+2q_2(1-q_2)-2q_1q_2 & \textit{when } \Gamma(t)=t^2\\ \mu-q_1(1-q_1)-2q_2(1-q_2)+2q_1q_2 & \textit{when } \Gamma(t)=2t-t^2\end{cases}$$

and

[5] Gini's mean difference was already used by von Andrae (1872) and Helmert (1876) as a measure of dispersion.

(2.12) $$\Delta_G(F) = q_0(1-q_0) + q_2(1-q_2) = q_1(1-q_1) + 2q_2(1-q_2) - 2q_1q_2 .$$

Note that Δ_G takes its maximum value when $q_0 = q_2 = \frac{1}{2}$.

The r dimensional case

Next, we consider the r dimensional case formed by the multinomial distribution of *r* deprivation indicators $X_1, X_2, ..., X_r$. In this case $\sum_{k=0}^{r} q_k = 1$ and the mean μ is given by (2.2). Similarly as in the 2x2 case we get that $D_\Gamma(F)$ admits the decomposition

(2.13) $$D_\Gamma(F) = \begin{cases} \mu + \Delta_\Gamma(F) & \textit{when } \Gamma \textit{ is convex} \\ \mu - \Delta_\Gamma(F) & \textit{when } \Gamma \textit{ is concave,} \end{cases}$$

where the dispersion measure $\Delta_\Gamma(F)$ is defined by

(2.14) $$\Delta_\Gamma(F) = \begin{cases} \sum_{k=0}^{r-1} \left[\sum_{j=0}^{k} q_j - \Gamma(\sum_{j=0}^{k} q_j) \right] & \textit{when } \Gamma \textit{ is convex} \\ \sum_{k=0}^{r-1} \left[\Gamma(\sum_{j=0}^{k} q_j) - \sum_{j=0}^{k} q_j \right] & \textit{when } \Gamma \textit{ is concave,} \end{cases}$$

Note that $D_\Gamma(F) \geq r - \sum_{k=0}^{r-1}\sum_{j=0}^{k} q_j = \mu$ and $\mu \leq D_\Gamma(F) \leq r$ when Γ *is convex*, and $0 \leq D_\Gamma(F) \leq \mu$ when Γ *is concave*. When Γ *is convex* the minimum value of $D_\Gamma(F)$ is attained when $\Delta_\Gamma(F) = 0$; i.e. when each individual of the population suffers from the same number of deprivations, whereas the maximum value of $D_\Gamma(F)$ is attained when $\Delta_\Gamma(F) = 0.5$; i.e. when 50 per cent of the population does not suffer from any deprivation and the remaining 50 per cent suffer from every dimension of deprivation. By contrast, for concave Γ the minimum and maximum values of $D_\Gamma(F)$ are attained when $\Delta_\Gamma(F)$ is equal to 0.5 and 0.

The decomposition (2.13) suggests that $D_\Gamma(F)$ obeys the principle of mean preserving spread when Γ is convex; i.e. $D_\Gamma(F)$ increases when the number of deprivations at the middle of the count distribution is shifted towards the tails, under the condition of fixed total number of deprivations. However, when Γ is concave, the summary measure $D_\Gamma(F)$ decreases as a consequence of a mean preserving spread. This is due to the fact that such an operation will increase the number of people who do not suffer from any deprivation and/or suffer from a few dimensions of deprivation.

As for the two-dimensional case, we get by inserting for $\Gamma(t) = t^2$ and $\Gamma(t) = 2t - t^2$ in (2.13) and (2.14) the following convenient expressions for the Gini measures of deprivation and dispersion, (2.15) $$D_\Gamma(F)=\begin{cases}\mu+\Delta_G(F) & when\ \Gamma(t)=t^2\\ \mu-\Delta_G(F) & when\ \Gamma(t)=2t-t^2.\end{cases}$$

where

(2.16) $$\Delta_G(F)=\sum_{k=0}^{r-1}kq_k(1-q_k)-2\sum_{j=0}^{r-1}\sum_{k=j+1}^{r-1}jkq_jq_k\ .$$

More generally, by inserting a parametric specification of Γ we can derive alternative parametric subfamilies of Δ and D. If the preference function is defined by

(2.17) $$\Gamma(t)=t^i\ ,$$

then

(2.18) $$\Delta_\Gamma(F)=\Delta_i(F)=\begin{cases}\sum_{k=0}^{r-1}\left[\sum_{j=0}^{k}q_j-(\sum_{j=0}^{k}q_j)^i\right], i\geq 1\\ \sum_{k=0}^{r-1}\left[(\sum_{j=0}^{k}q_j)^i-\sum_{j=0}^{k}q_j\right], 0<i\leq 1.\end{cases}$$

Note that Δ_i can be considered as a measure of left-spread when $0<i<1$ and a measure of right-spread when $i>1$. The next sub-section will clarify the relationship between a mean preserving spread, second-degree upward and downward count distribution dominance and association rearrangements.

3. Association rearrangements

To provide a normative justification of upward and downward count distribution dominance as well as for employing the deprivation measures D_Γ for concave and convex Γ, we introduce association intervention principles similar to those discussed by Epstein and Tanny (1980), Boland and Proschan (1988) and Tsui (1999, 2002). We will analyze rearrangements in the achievement space that lead to mean-preserving spreads/contractions in the space of the deprivation counts. The previous literature does not distinguish between positive and negative association (or correlation). By contrast, we make a distinction between whether an association rearrangement comes from a distribution characterized by positive or negative association between two or several deprivation indicators, in the spirit of the statistical literature on measurement of association in multidimensional contingency tables (formed by two or several dicotomous variables). Various authors (see e.g. Yule, 1910 and Mosteller, 1968) have emphasized the importance of separating the information of a 2x2 table provided by the association between the social indicators X_1 and X_2 from the information provided by the marginal distributions (p_{0+},p_{1+}) and (p_{+0},p_{+1}). For 2x2 tables (see Table 3.1) this objective corresponds to introducing measures of association that are invariant under the transformation

(3.1) $$p_{ij}\rightarrow a_ib_jp_{ij}$$

for any set of positive numbers $\{a_i\}$ and $\{b_j\}$ such that $\sum_{i=0}^{1}\sum_{j=0}^{1}a_ib_jp_{ij}=1$.

The cross-product α introduced by Yule (1900) and defined by

(3.2) $$\alpha = \frac{p_{00}\, p_{11}}{p_{01}\, p_{10}},$$

is a measure of association that satisfies the invariance condition (3.1), whereas the correlation coefficient does not. Thus, the association measure α and the marginal distributions (p_{0+}, p_{1+}) and (p_{+0}, p_{+1}) together provide complete information of Table 3.1. Note that $\alpha \in [0,\infty)$, $\alpha = 1$ if the indicators X_1 and X_2 are independent, $\alpha = 0$ when $p_{00} = p_{11} = 0$ and $\alpha \to \infty$ when $p_{01} = p_{10} = 0$. In the former case there is perfect negative association between the two indicators, whereas it is perfect positive association in the latter case. Accordingly, it is required to make a distinction between positive association increasing rearrangements, positive association decreasing rearrangements, negative association increasing rearrangements and negative association decreasing rearrangements[6].

DEFINITION 3.1A. *Consider a 2x2 table with parameters* $(p_{00}, p_{01}, p_{10}, p_{11})$ *where* $\sum\sum p_{ij} = 1$ *and* $\alpha > 1$. *The following marginal-free change* $(p_{00}+\delta, p_{01}-\delta, p_{10}-\delta, p_{11}+\delta)$ *is said to provide marginal distributions preserving positive association increasing (decreasing) rearrangement if* $\delta > 0$ $(\delta < 0)$.

DEFINITION 3.1B. *Consider a 2x2 table with parameters* $(p_{00}, p_{01}, p_{10}, p_{11})$ *where* $\sum\sum p_{ij} = 1$ *and* $\alpha < 1$. *The following marginal-free change* $(p_{00}+\delta, p_{01}-\delta, p_{10}-\delta, p_{11}+\delta)$ *is said to provide a marginal distributions preserving negative association increasing (decreasing) rearrangement if* $\delta < 0$ $(\delta > 0)$.

An illustration is provided in the tables below, where the right (left) panel of Table 3.1 is obtained from the left (right) panel by a positive association increasing (decreasing) rearrangement, whereas the right (left) panel of Table 3.2 can be obtained from the left (right) panel by a negative association increasing (decreasing) rearrangement.

Table 3.1. Rearrangement that increases a positive association

	0	1				0	1	
0	.30	.20	.50		0	.31	.19	.50
1	.20	.30	.50		1	.19	.31	.50
	.50	.50	1			0.50	0.50	1

Table 3.2. Rearrangement that increases a negative association

	0	1				0	1	
0	.20	.30	.50		0	.19	.31	.50
1	.30	.20	.50		1	.31	.19	.50
	.50	.50	1			.50	.50	1

[6] For similar definitions of association increasing rearrangements based on the correlation coefficient we refer to Atkinson and Bourguignon (1982), Dardanoni (1995) , Tsui (1999, 2002), Bourguignon and Chakravarty (2003), Duclos et al. (2006) and Kakwani and Silber (2008). See also Tchen (1980) who deals with positive association (or concordance) between bivariate probability measures and Decancq (2011) for a recent generalization of these principles and an analysis of their links with stochastic dominance.

Mean preserving association rearrangements

The association increasing/decreasing rearrangement principles defined by Definitions 3.1A and 3.1B prove to support second-degree downward/upward dominance under the condition of unchanged marginal distributions; i.e. the number of people suffering from each of the deprivation indicators are kept fixed. However, since real world interventions normally concern trade-offs that allow reduction in one deprivation indicator at the cost of a rise in another deprivation indicator, we find it attractive to introduce association increasing/decreasing rearrangement principles that rely on the condition of fixed number of total deprivations, rather than on the condition of keeping the number suffering from each of the indicators fixed. Since the correlation coefficient does not satisfy the invariance condition (3.1) it is not fully informative about the association between two variables, and consequently inappropriate as a measure of association for defining mean preserving increasing (decreasing) rearrangement principles. This limitation of the correlation coefficient motivates our use of the cross-product α as a measure of association in the definitions of the principles of mean preserving increasing (decreasing) rearrangement as well as in Definitions 3.1A and 3.1B, although the condition of fixed marginal distributions allows the use of the correlation coefficient in the latter definitions.

DEFINITION 3.2A. *Consider a 2x2 table with parameters* $(p_{00}, p_{01}, p_{10}, p_{11})$ *where* $\sum\sum p_{ij} = 1$ *and* $\alpha > 1$. *The following change* $(p_{00}+\delta, p_{01}, p_{10}-2\delta, p_{11}+\delta)$ *is said to provide a mean preserving positive association increasing (decreasing) rearrangement if* $\delta > 0$ $(\delta < 0)$.

DEFINITION 3.2B. *Consider a 2x2 table with parameters* $(p_{00}, p_{01}, p_{10}, p_{11})$ *where* $\sum\sum p_{ij} = 1$ *and* $\alpha < 1$. *The following free change* $(p_{00}+\delta, p_{01}, p_{10}-2\delta, p_{11}+\delta)$ *is said to provide a mean preserving negative association increasing (decreasing) rearrangement if* $\delta < 0$ $(\delta > 0)$.

It follows straightforward from Definitions 3.2A and 3.2B that the mean preserving association principles make a mean preserving rearrangement that reduces the number of people suffering from indicator X_1 at the cost of increasing the number of people suffering from indicator X_2 when $\delta > 0$ and vice versa when $\delta < 0$. As illustrated by Table 3.3 the right (left) panel can be obtained from the left (right) panel by a mean preserving positive increasing (decreasing) rearrangement, since the association is negative and the mean is kept fixed equal to 1 under the rearrangement where $\delta = .01$.

Table 3.3. Illustration of mean preserving decreasing negative association rearrangements

	0	1				0	1	
0	.20	.30	.50		0	.21	.30	.51
1	.30	.20	.50		1	.28	.21	.49
	.50	.50	1			.49	.51	1

Since the condition of fixed marginal distributions also implies that the means are kept fixed, it follows that Definitions 3.1A and 3.1B can be considered as a special case of Definitions 3.2A and 3.2B, respectively. Thus, we will focus attention on Definitions 3.2A and 3.2B below.

Definitions 3.2A and 3.2B can readily be extended to higher dimensions. However, for a large number of dimensions the standard subscript notation becomes cumbersome. Thus, we find it convenient to introduce the following simplified subscript notation p_{ijk} , where *i* and *j* represents two arbitrary chosen deprivation dimensions and *m* represents the remaining *r-2* dimensions and α_{ijm} is defined by

(3.3)
$$\alpha_{ijm} = \frac{p_{iim} p_{jjm}}{p_{ijm} p_{jim}} ,$$

where *m* is a r-2 dimensional vector of any combination of zeroes and ones. In this case association is defined by *r(r-1)/2* cross-products.

In order to deal with *r*-dimensional counting data we introduce the following generalization of Definitions 2.2A and 2.2B,

DEFINITION 3.3A. *Consider a 2x2x…x2 table formed by s dichotomous variables with parameters* $(p_{iim}, p_{ijm}, p_{jim}, p_{jjm})$ *where* $\sum\sum\sum p_{ijm} = 1$ *and* $\alpha_{ijm} > 1$. *The following change* $(p_{iim} + \delta, p_{ijm}, p_{jim} - 2\delta, p_{jjm} + \delta)$ *is said to provide a mean preserving positive association increasing (decreasing) rearrangement if* $\delta > 0$ $(\delta < 0)$.

DEFINITION 3.3B. *Consider a 2x2x…x2 table formed by s dichotomous variables with parameters* $(p_{iim}, p_{ijm}, p_{jim}, p_{jjm})$ *where* $\sum\sum\sum p_{ijm} = 1$ *and* $\alpha_{ijm} < 1$. *The following change* $(p_{iim} + \delta, p_{ijm}, p_{jim} - 2\delta, p_{jjm} + \delta)$ *is said to provide a mean preserving negative association increasing (decreasing) rearrangement if* $\delta < 0$ $(\delta > 0)$.

As is demonstrated by Theorems 3.1A below, a social planner who is in favour of second-degree downward dominance will consider a mean preserving positive association increasing rearrangement as well as a mean preserving negative association decreasing rearrangement as a rise in overall deprivation. By contrast, a planner who favours upward second-degree dominance will consider such rearrangement as a reduction in the overall deprivation. Moreover, it is proved that the principles of mean preserving association increasing/decreasing rearrangement are equivalent to the mean preserving spread/contraction defined by

DEFINITION 3.4. *Let* F_1 *and* F_2 *be members of the family* ***F*** *of count distributions based on s deprivation indicators and where* F_1 *and* F_2 *are assumed to have equal means. Then* F_2 *is said to differ from* F_1 *by mean preserving spread (contraction) if* $\Delta_\Gamma(F_2) > \Delta_\Gamma(F_1)$ *for all convex* Γ $(\Delta_\Gamma(F_2) < \Delta_\Gamma(F_1)$ *for all concave* Γ *).*

Note that Definition 3.4 is equivalent to a sequence of the mean preserving spread introduced by Rothschild and Stiglitz (1970). This is easily seen by combining statements (ii) and (iii) of Theorem 3.1A and equation (A5) of the Appendix.

THEOREM 3.1A. *Let F_1 and F_2 be members of the family **F** of count distributions based on s deprivation indicators and assume that F_1 and F_2 have equal means. Then the following statements are equivalentt*

(i) F_1 second-degree downward dominates F_2
(ii) F_2 can be obtained from F_1 by a sequence of mean preserving positive association increasing rearrangements when $\alpha > 1$ and a sequence of mean preserving negative association decreasing rearrangements when $\alpha < 1$
(iii) F_2 can be obtained from F_1 by a mean preserving spread.

(Proof in Appendix).

THEOREM 3.1B. *Let F_1 and F_2 be members of the family **F** of count distributions based on s deprivation indicators and assume that F_1 and F_2 have equal means. Then the following statements are equivalent*

(i) F_1 second-degree upward dominates F_2
(ii) F_2 can be obtained from F_1 by a sequence of mean preserving positive association decreasing rearrangements when $\alpha > 1$ and a sequence of mean preserving negative association increasing rearrangements when $\alpha < 1$.
(iii) F_2 can be obtained from F_1 by a mean preserving contraction

(Proof in Appendix).

By combining Theorems 2.2A and 3.1A and Theorems 2.2B and 3.1B it follows that the D_Γ-measures satisfy the association intervention principles introduced above, where a distinction has been made between whether an association rearrangement comes from a distribution characterized by positive or negative association.

4. Accounting for different weights

Replacing the outcome 1 by the weights w_1 and w_2 as outcomes for the marginal indicator distributions in the two-dimensional case, the distribution of deprivation for two dimensions is given by the following table

Table 4.1. The distribution of weighted deprivation in two dimensions

	$\tilde{X}_1$			
$\tilde{X}_2$		0	w_2	
	0	p_{00}	p_{01}	p_{0+}
	w_1	p_{10}	p_{11}	p_{1+}
		p_{+0}	p_{+1}	1

Next, by assuming that $w_1 \le w_2$, the variable $\tilde{X}$ defined by $\tilde{X} = \tilde{X}_1 + \tilde{X}_2 = w_1 X_1 + w_2 X_2$ can be considered as a weighted counting variable. The distribution $\tilde{F}$ of $\tilde{X}$ is given by

(4.1) $$\tilde{F}(z)=\begin{cases} p_{00} & if\ \ z=0 \\ p_{00}+p_{10} & if\ \ z=w_1 \\ p_{00}+p_{10}+p_{01} & if\ \ z=w_2 \\ 1 & if\ \ z=w_1+w_2. \end{cases}$$

Theorem 2.1 shows that a social planner who supports Axioms 1 – 4 will rank count distributions of deprivation according to the criterion D_Γ defined by

(4.2) $$\tilde{D}_\Gamma(\tilde{F})=\int(1-\Gamma(\tilde{F}(z)))dz,$$

where Γ, with $\Gamma(0)=0$ and $\Gamma(1)=1$, is *a non-decreasing* function that represents the preferences of the social planner. Thus, the social planer considers the distribution $\tilde{F}$ that minimizes $\tilde{D}_\Gamma(\tilde{F})$ to be the most favorable among those being compared. Since $\tilde{F}$ denotes the weighted count variable distribution of deprivation, $\tilde{D}_\Gamma(\tilde{F})$ can be considered as a measure of the extent of deprivation exhibited by the distribution $\tilde{F}$. Now, by inserting the mean $\tilde{\mu}=\int(1-\tilde{F}(z))dz$ in (4.2) we obtain the following decomposition

(4.3) $$\tilde{D}_\Gamma(\tilde{F})=\begin{cases} \tilde{\mu}+\tilde{\Delta}_\Gamma(\tilde{F}) & when\ \Gamma\ is\ convex \\ \tilde{\mu}-\tilde{\Delta}_\Gamma(\tilde{F}) & when\ \Gamma\ is\ concave \end{cases}$$

where $\tilde{\Delta}_\Gamma(\tilde{F})$ is defined by

(4.4) $$\tilde{\Delta}_\Gamma(\tilde{F})=\begin{cases} \int\left(\tilde{F}(z)-\Gamma(\tilde{F}(z))\right)dz & when\ \Gamma\ is\ convex \\ \int\left(\Gamma(\tilde{F}(z))-\tilde{F}(z)\right)dz & when\ \Gamma\ is\ concave. \end{cases}$$

Expressions (4.3) and (4.4) demonstrate that Theorems 2.2A, 2.2 B, 4.1A and 4.1B are valid for weighted count distributions as well.

5. Summary and discussion

The conventional approach in official statistics as well as in most empirical studies of multidimensional deprivation is focusing on the distribution of the number of dimensions in which people suffer from deprivation. This paper is concerned with the problem of ranking and quantifying the extent of deprivation exhibited by multidimensional distributions of deprivation where the multiple attributes in which an individual can be deprived are represented by dichotomized variables. By drawing on the rank-dependent social evaluation framework that originates from Sen (1974) and Yaari (1988) this paper introduces summary measures of deprivation that proves to allow decomposition into extent of and dispersion in the distribution of multiple deprivations. To provide a normative justification of the proposed deprivation measures two intervention principles affecting the association (correlation) between the different deprivation indicators and the spread of the deprivation counts are adopted.

Notice that that the deprivation indicators are assumed to be perfect substitutes by construction, since the counting approach attaches an equal weight to each of the single indicators. As is demonstrated in Section 4, the framework provided in this paper can be extended to allow for different weighting profiles across the multidimensional distribution of deprivations.

Appendix - *Proofs*

LEMMA 1. *Let H be the family of bounded, continuous and non-negative functions on [0,1] which are positive on* $\langle 0,1\rangle$ *and let g be an arbitrary bounded and continuous function on* $[0,1]$. *Then*

$$\int g(t)h(t)dt > 0 \quad \text{for all } h \in H$$

implies

$$g(t) \geq 0 \quad \text{for all } t \in [0,1]$$

and the inequality holds strictly for at least one $t \in \langle 0,1\rangle$.

Proof of Theorems 2.2A and 2.2B. Using integration by parts, we get:

$$D_\Gamma(F_2) - D_\Gamma(F_1) = \int_0^1 (1-\Gamma(t))d(F_2^{-1}(t) - F_1^{-1}(t))$$

$$= -\Gamma'(0)\int_0^1 (F_2^{-1}(t) - F_1^{-1}(t))dt + \int_0^1 \Gamma''(u)\int_u^1 (F_2^{-1}(t) - F_1^{-1}(t))dtdu.$$

Thus, if (i) holds then $D_\Gamma(F_1) < D_\Gamma(F_2) \quad \text{for all } \Gamma \in \Omega_1$.

To prove the converse statement we restrict to preference functions $\Gamma \in \Omega_1$. Hence,

$$D_\Gamma(F_2) - D_\Gamma(F_1) = \int_0^1 \Gamma''(u)\int_u^1 (F_2^{-1}(t) - F_1^{-1}(t))dtdu,$$

and the result is obtained by applying Lemma 1.

The proof of Theorem 2.2B is analogous to the proof of Theorem 2.2A, and is based on the expression

$$D_\Gamma(F_2) - D_\Gamma(F_1) = \int_0^1 (1-\Gamma(t))d(F_2^{-1}(t) - F_1^{-1}(t))$$

$$= -\Gamma'(1)\int_0^1 (F_2^{-1}(t) - F_1^{-1}(t))dt - \int_0^1 \Gamma''(u)\int_0^u (F_2^{-1}(t) - F_1^{-1}(t))dtdu,$$

which is obtained by using integration by parts. Thus, by using arguments like those in the proof of Theorem 2.2A the results of Theorem 2.2B are obtained.

Proof of Theorems 4.1A and 4.1B.

As demonstrated by Hardy, Littlewood and Polya (1934) an equivalent condition of Definition 2.1A is given by

(A1) $$\int_y^\infty F(x)dx \geq \int_y^\infty \tilde{F}(x)dx \quad \text{for all } y \in [0,\infty),$$

where $F(k) = F_1(k) = \sum_{j=0}^{k} q_j \quad and \quad \tilde{F}(k) = F_2(k) = \sum_{j=0}^{k} \tilde{q}_j$.

By inserting for *F* and $\tilde{F}$ in (A1) we get that *F* second-degree downward dominates $\tilde{F}$ if and only if

(A2) $$\sum_{j=i}^{r-1}\sum_{k=0}^{j} q_k \geq \sum_{j=i}^{r-1}\sum_{k=0}^{j} \tilde{q}_k \quad \text{for } i = 0,1,...,r-1.$$

Next, assume that (ii) is true; i.e.
$\tilde{p}_{iim} = p_{iim} + \delta, \tilde{p}_{ijm} = p_{ijm}, \tilde{p}_{jim} = p_{jim} - 2\delta$ *and* $\tilde{p}_{jjm} = p_{jjm} + \delta$ which we assume corresponds to changes in the number of people suffering from *t* (p_{iim}), *t+1* $(p_{ijm} + p_{jim})$ and *t+2* (p_{jjm}) deprivations such that

(A3) $$\tilde{q}_t = q_t + \delta, \tilde{q}_{t+1} = q_{t+1} - 2\delta, \tilde{q}_{t+2} = q_{t+2} + \delta \text{ and } \tilde{q}_k = q_k \text{ for all } k \neq t, t+1, t+2,$$

which means that the mean of $\tilde{F}$ is equal to the mean of *F*.

Inserting for (A3) in $\tilde{F}$ yields

(A4) $$\tilde{F}(k)=\sum_{j=0}^{k}\tilde{q}_j=\begin{cases}\sum_{j=0}^{k}q_j & for\ k=0,1,...,t-1\\ \sum_{j=0}^{k}q_j+\delta & for\ k=t\\ \sum_{j=0}^{k}q_j-\delta & for\ k=t+1\\ \sum_{j=0}^{k}q_j & for\ k=t+2,t+3,...,r.\end{cases}$$

It follows by straightforward calculations that (A4) implies (A2) and thus that (ii) implies (i).

To prove the converse statement, assume that (i) is true, i.e. that (A2) is valid. Since F and $\tilde{F}$ are step functions it can be demonstrated that there exists a sequence of discrete distribution functions $F_0^*, F_1^*,...,F_s^*$ such that $F=F_0^*$, $\tilde{F}=F_s^*$ and F_{i+1}^* differs from F_i^* by a mean preserving positive association increasing rearrangement, i.e. $F_{i+1}^*-F_i^*$ is given by

(A5) $$F_{i+1}^*(k)-F_i^*(k)=\begin{cases}0 & for\ k=0,1,...,t-1\\ \delta & for\ k=t\\ -\delta & for\ k=t+1\\ 0 & for\ k=t+2,t+3,...,r.\end{cases}$$

Next, we use (A5) to construct F_1^* from F, F_2^* from F_1^* and finally $\tilde{F}$ from F_{s-1}^*. The required number of iterations (s) depends on the number of steps exhibited by the difference $\tilde{F}-F$.

The equivalence between (i) and (iii) follows directly from Theorem 2.2 A.

The proof of Theorem 4.1B is analogous to the proof of Theorem 4.1A. Thus, by using arguments like those in the proof of Theorem 4.1A the results of Theorem 4.1B are obtained.

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